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THE INFLUENCE OF RELATIVE VELOCITY ON THE EDDY STRUCTURE BETWEEN TWO SPHERES IN STOKES FLOW

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Abstract—Davis *et al.* (1976) have shown that if two solid spheres move together in an axisymmetric Stokes flow, then provided they are sufficiently close, a body of fluid becomes trapped between the spheres. Here it is shown how the small eddy motions induced in this trapped fluid are significantly disrupted when one sphere moves relative to the other.

1. INTRODUCTION

Although Stokes flow cannot separate from a solid sphere in isolation, Davis *et al.* (1976) showed that if the centres of two equal spheres are less than 3.57 radii apart in an axisymmetric flow, then viscous wakes develop and, as the spheres are placed closer together, toroidal vortices are successively established in a nested manner around the axis of symmetry. A similar result holds for unequal spheres, the analysis being a special case of what follows in this paper. The practical significance lies in the outermost and strongest eddy whose outer boundary separates fluid in the external flow from the body of fluid which becomes trapped between the spheres. Since the boundaries of the eddies are determined by the zeros of exponentially small terms, the structure described above is, except for spheres in contact, difficult to reproduce experimentally. This paper examines the effect on the eddy structure of one sphere moving relative to the other, both slowly enough and at sufficient distance for the quasi-static approximation described by Brenner (1961) and Rushton & Davies (1973) to be valid. At small distances, the method of matched asymptotic expansions employed by O'Neill & Majumdar (1970) is appropriate.

As in the previously mentioned work, axes are fixed with respect to the sphere from which separation is sought. Sphere II is assumed to move at speed ϵ against a uniform stream of unit speed and towards sphere I which is at rest. Separation on sphere I is delayed or enhanced according as ϵ is positive or negative. In the former case, it is found that for large enough ϵ (0.05 for equal spheres), separation occurs not through a toroidal vortex forming on sphere I, but because the fluid trapped instantaneously around sphere II has spread as far as the stationary sphere.

It should be noted that in practical applications, ϵ is likely to be related to other physical variables. However the available formulae for this dependence lose accuracy as the spheres approach each other, which situation is not yet fully understood. This paper, in which the spheres are about a radius apart, is hopefully a contribution to its understanding.

2. STATEMENT AND SOLUTION OF THE PROBLEM

Two rigid spheres are placed in a steady stream of infinite incompressible viscous fluid of constant density and viscosity, so that the line of centres of the spheres, one of which is held at rest, is parallel to the direction of the stream.

Choosing the line of centres to be the axis of cylindrical polar coordinates (r, θ, z), bispherical coordinates are defined by

$$r = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}. \tag{2.1}$$

The external fluid region is then $-\xi_2 \le \xi \le \xi_1$, $0 \le \eta \le \pi$, where, with the spheres I and II having radii 1 and b respectively,

$$c = \sinh \xi_1, \quad b = c \operatorname{cosech} \xi_2 = \sinh \xi_1 / \sinh \xi_2 . \tag{2.2}$$

The distance D between the centres of the spheres is given by

$$D = c(\coth \xi_1 + \coth \xi_2) = \frac{\sinh(\xi_1 + \xi_2)}{\sinh \xi_2}, \tag{2.3}$$

and the coordinate values ξ_1, ξ_2 are obtained directly from b, D by the equations

$$\cosh \xi_1 = \frac{D^2 + 1 - b^2}{2D}, \quad \cosh \xi_2 = \frac{D^2 - 1 + b^2}{2bD}.$$

Evidently the fluid velocity has cylindrical components $(u, 0, w)$ which are independent of θ and given by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{2.4}$$

where the stream function ψ satisfies in the Stokes approximation the equation

$$\Lambda^4 \psi = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0 . \tag{2.5}$$

Taking the stream to be in the negative z-direction, the condition at infinity is

$$\psi \sim \frac{1}{2} r^2 \quad \text{as} \quad r^2 + z^2 \to \infty . \tag{2.6}$$

Zero velocity on sphere I requires that

$$\psi = \frac{\partial \psi}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_1 \tag{2.7}$$

whilst the no-slip condition on sphere II, which moves with speed ϵ in the positive z direction, yields

$$\psi + \frac{1}{2}\epsilon r^2 = \frac{\partial}{\partial \xi}\left(\psi + \frac{1}{2}\epsilon r^2\right) = 0 \quad \text{at} \quad \xi = -\xi_2 . \tag{2.8}$$

The appropriate solution of [2.5] is of the form

$$\begin{aligned} \psi &= \frac{c^2(1-\sigma^2)}{2(\cosh \xi - \sigma)^2} + c^2(2\cosh \xi - 2\sigma)^{-3/2} \sum_1^\infty U_n(\xi) V_n(\sigma) \\ &= c^2(2\cosh \xi - 2\sigma)^{-3/2} \chi \end{aligned} \tag{2.9}$$

where

$$\sigma = \cos\eta, \quad V_n(\sigma) = P_{n-1}(\sigma) - P_{n+1}(\sigma)$$

$$U_n(\xi) = W_n(\nu) = A_n \cosh(\beta_n - 1)\nu + B_n \sinh(\beta_n - 1)\nu + C_n \cosh(\beta_n + 1)\nu + D_n \sinh(\beta_n + 1)\nu$$

$$\beta_n = n + \frac{1}{2}, \quad \xi = s + \nu, \quad s = \frac{1}{2}(\xi_1 - \xi_2), \quad q = \frac{1}{2}(\xi_1 + \xi_2).$$

Employing the expansion, valid for $\xi \neq 0$:

$$\frac{4(1-\sigma^2)}{(2\cosh\xi - 2\sigma)^{1/2}} = \sum_1^{\infty} K_n(\xi) V_n(\sigma)$$

where

$$K_n(\xi) = \frac{2\left(\beta_n^2 - \frac{1}{4}\right)}{\beta_n(\beta_n^2 - 1)} e^{-\beta_n|\xi|}(\beta_n \sinh|\xi| + \cosh\xi),$$

the four conditions [2.7], [2.8] can be written

$$W_n(q) + \frac{1}{2}K_n(\xi_1) = 0,$$

$$W_n'(q) + \frac{1}{2}K_n'(\xi_1) = 0,$$

$$W_n(-q) + \frac{1}{2}(1+\epsilon)K_n(-\xi_2) = 0,$$

$$W_n'(-q) + \frac{1}{2}(1+\epsilon)K_n'(-\xi_2) = 0. \qquad [2.10]$$

By addition and subtraction, these four equations split into disjoint pairs which determine, for each $n \geq 1$, the coefficients A_n, C_n and B_n, D_n. In the particular case $s = \epsilon = 0$, the coefficients given by Davis *et al.* (1976) are, except for a scaling factor, recovered. The extension of that paper to unequal spheres is obtained by setting $\epsilon = 0$ with $s \neq 0$.

3. SINGLE SPHERE MOVING AGAINST STREAM

Since we are considering what happens to the flow pattern as sphere II approaches sphere I, it is of interest to know what it looks like when the spheres are far apart and in particular what flow pattern exists instantaneously around a single unit sphere moving against the stream. In terms of spherical polar coordinates (R, ϑ, θ) with origin at the centre of the sphere, the operator Λ^2 of [2.5] has the form

$$\Lambda^2 = \frac{\partial^2}{\partial R^2} + \frac{\sin\vartheta}{R^2}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\right)$$

and the stream function is given by

$$\psi = \frac{\sin^2\vartheta}{4R}[(R-1)^2(2R+1) - \epsilon(3R^2-1)] \quad (R \geq 1). \qquad [3.1]$$

Since Stokes flows are reversible, the minus sign of a negative ϵ can be accounted for by changing the direction of the basic streaming flow. Then sphere II always moves towards sphere I but does so with or against the stream according as ϵ is negative or positive. The instantaneous flow patterns determined by [3.1] are illustrated in figure 1. Defining $R_0(\epsilon)$ to be the value of R at which $\psi = 0$ for $\epsilon > 0$, table 1 shows how R_0 is sensitive to small changes in ϵ from 0.

Table 1.

ϵ	0.03	0.05	0.1	0.2
R_0	1.167	1.225	1.347	1.550

This suggests that the introduction of even a small relative velocity is likely to have considerable impact on the eddy structure described by Davis *et al.* (1976).

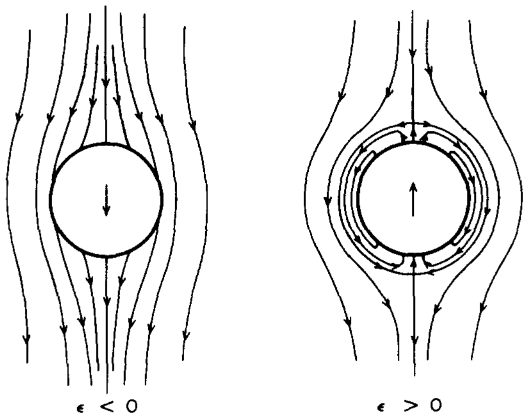

Figure 1. The instantaneous flow patterns past a single moving sphere.

4. SEPARATION OF THE FLOW FROM THE STATIONARY SPHERE

Substituting [2.1], [2.9] into [2.4], the axial velocity is given by

$$w(\xi, s) = -1 + \cosh \frac{1}{2} \xi \sum_1^\infty (-1)^n \beta_n U_n(\xi) . \qquad [4.1]$$

Separation from sphere I begins at $\eta = \pi$ when the derivative of the axial velocity at $\xi = \xi_1$ changes sign. The condition $(\partial w/\partial \xi)(\xi_1, s) = 0$ implies, on substituting [2.9]:

$$\sum_1^\infty (-1)^n \beta_n \left[W_n''(q) + \frac{1}{2} K_n''(\xi_1) \right] = 0$$

which, after eliminating the coefficients A_n, B_n, C_n, D_n can be written

$$\sum_1^\infty (-1)^n \beta_n \left(\beta_n^2 - \frac{1}{4}\right) \left\{ \frac{e^{\beta_n \xi_2} \sinh \xi_1 - e^{-\beta_n \xi_1} \sinh \xi_2 + (1+\epsilon)[e^{\beta_n \xi_1} \sinh \xi_2 - e^{-\beta_n \xi_2} \sinh \xi_1]}{\sinh 2\beta_n q + \beta_n \sinh 2q} \right.$$

$$\left. + \frac{e^{\beta_n \xi_2} \sinh \xi_1 + e^{-\beta_n \xi_1} \sinh \xi_2 - (1+\epsilon)[e^{\beta_n \xi_1} \sinh \xi_2 + e^{-\beta_n \xi_2} \sinh \xi_1]}{\sinh 2\beta_n q - \beta_n \sinh 2q} \right\} = 0 .$$

Applying the usual Watson transformation (Davis & O'Neill 1977), dividing by $\sinh \xi_1 \sinh \xi_2$ in virtue of [2.2] and retaining only the leading residue terms, this becomes

$$\epsilon \int_0^\infty \frac{y\left(y^2+\frac{1}{4}\right)}{\cosh \pi y}\left\{\frac{b \cos y\xi_2 - \cos y\xi_1}{\sin 2yq + y \sinh 2q} + \frac{b \cos y\xi_2 + \cos y\xi_1}{\sin 2yq - y \sinh 2q}\right\} \mathrm{d}y$$

$$+ \frac{\pi}{4q^3} Re \left[\frac{\sigma_1(\sigma_1^2+q^2)\left\{(2+\epsilon)\left[\sin\frac{\sigma_1\xi_1}{2q} + b \sin\frac{\sigma_1\xi_2}{2q}\right] + i\epsilon\left[\cos\frac{\sigma_1\xi_1}{2q} - b\cos\frac{\sigma_1\xi_2}{2q}\right]\right\}}{\cosh\frac{\pi\sigma_1}{2q}(2q\cos\sigma_1 + \sinh 2q)} \right.$$

$$\left. - \frac{\tau_1(\tau_1^2+q^2)\left\{(2+\epsilon)\left[\sin\frac{\tau_1\xi_1}{2q} - b \sin\frac{\tau_1\xi_2}{2q}\right] + i\epsilon\left[\cos\frac{\tau_1\xi_1}{2q} + b\cos\frac{\tau_1\xi_2}{2q}\right]\right\}}{\cosh\frac{\pi\tau_1}{2q}(2q\cos\tau_1 - \sinh 2q)} \right] \sim 0. \qquad [4.2]$$

where, as defined by Davis & O'Neill (1977), σ_1 and τ_1 are respectively the dominant complex roots of $2q \sin z \pm z \sinh 2q = 0$ in the first quadrant. The asymptotic approximations:

$$\sigma_m \sim \left(2m - \frac{1}{2}\right)\pi + \left[i - \frac{1}{\left(2m-\frac{1}{2}\right)\pi}\right] \log_e\left[(4m-1)\pi\frac{\sinh 2q}{2q}\right]$$

$$\tau_m \sim \left(2m + \frac{1}{2}\right)\pi + \left[i - \frac{1}{\left(2m+\frac{1}{2}\right)\pi}\right] \log_e\left[(4m+1)\pi\frac{\sinh 2q}{2q}\right]$$

can be used for iterative computation of σ_1, τ_1 for various q and some values are listed in table 2. A much more comprehensive list for the similar equations $\alpha \sin z \pm z \sin \alpha = 0$ is given by Wakiya (1975). The factors $(2+\epsilon)$ and ϵ in [4.2] can be respectively identified as the sum and difference of the sphere velocities relative to the stream. Table 3 shows solutions q of [4.2] with corresponding D (given by [2.3]) for various values of ϵ and b. The $\epsilon = 0$, $b = 1$ solution is that given by Davis *et al.* (1976) and is included for completeness. Note that although the geometry of cases $b = 0.5$, 2 is similar, D is always measured in terms of the radius of sphere I and so there is a scaling factor of 2 to be taken into account when assessing the list of values of D for $b = 0.5$, 2. In particular, for $\epsilon = 0$, it can be observed that since $2.617 \times 2 > 5.131$, separation

Table 2.

$2q$	σ_1	τ_1
0.2	4.21104 + 2.25796i	7.49686 + 2.77559i
0.4	4.20700 + 2.27955i	7.49443 + 2.79620i
0.6	4.20035 + 2.31516i	7.49041 + 2.83019i
0.8	4.19118 + 2.36427i	7.48488 + 2.87707i
1.0	4.17963 + 2.42621i	7.47791 + 2.93617i
1.2	4.16589 + 2.50021i	7.46960 + 3.00675i
1.4	4.15014 + 2.58544i	7.46005 + 3.08796i
1.6	4.13260 + 2.68104i	7.44938 + 3.17897i
1.8	4.11351 + 2.78619i	7.43771 + 3.27892i
2.0	4.09310 + 2.90008i	7.42516 + 3.38701i
2.2	4.07163 + 3.02196i	7.41183 + 3.50247i
2.4	4.04932 + 3.15111i	7.39784 + 3.62460i
2.6	4.02640 + 3.28689i	7.38329 + 3.75275i
2.8	4.00309 + 3.42868i	7.36827 + 3.88635i
3.0	3.97958 + 3.57596i	7.35288 + 4.02487i

Table 3.

	$b = 0.5$		$b = 1.0$		$b = 2.0$	
ϵ	$2q$	D	$2q$	D	$2q$	D
−0.2	2.552	2.775	2.436	3.675	2.378	5.171
−0.1	2.482	2.697	2.402	3.624	2.368	5.151
−0.05	2.443	2.655	2.384	3.597	2.363	5.141
0	2.408	2.617	2.367	3.572	2.358	5.131
0.01			2.363	3.566		
0.03			2.356	3.556		
0.05	2.369	2.577	2.349	3.546	2.352	5.120
0.1	2.327	2.534	2.330	3.518	2.347	5.109
0.2	2.229	2.439	2.291	3.462	2.336	5.086

begins first on the larger sphere as the two are placed closer together. This is in agreement with the results of Davis & O'Neill (1977) for the sphere at rest near a plane.

Table 3 shows that D is a decreasing function of ϵ which indicates that a receding sphere has a sucking effect which encourages the fluid to separate from the stationary sphere whilst an approaching sphere has a squeezing effect which tends to suppress this separation. Alternatively, interpreting negative ϵ as a reversal of the basic flow, the fluid is deflected towards or away from the axis $\eta = \pi$ according as sphere II approaches sphere I against or with the stream. Further, it is observed that the smaller the value of b, the more rapidly does D change with ϵ. Indeed there is an ϵ of ≈ 0.07 for which separation occurs at exactly the same geometry in the $b = 0.5$, 2 cases. Here the squeezing effect of the small sphere approaching the large sphere is sufficiently greater than that of vice-versa as to exactly cancel the fact that when both spheres are at rest, separation occurs on the larger sphere at greater spacing than for the smaller sphere.

The dependence of D on b and ϵ is at variance with what might have been anticipated from Section 3. For a single sphere with $\epsilon > 0$, a body of fluid is essentially trapped between $R = 1$ and $R = R_0(\epsilon)$. When the sphere moves against the stream towards a sphere at rest, although not strictly dynamically equivalent, one might expect the moving sphere to have the effect of a larger solid sphere, enhancing the onset of separation on the fixed spheres. The results of table 3 show this type of physical argument to be erroneous. However, a question which emerges from this consideration of the single sphere is whether, for $\epsilon > 0$, separation occurs on sphere I with the formation of a toroidal eddy as for the $\epsilon = 0$ case (Davis *et al.* 1976) or because the body of fluid effectively attached to sphere II has spread to the surface of sphere I. Evidently the former possibility will occur for small enough ϵ whilst it can be anticipated that above a critical value of ϵ, the latter will be the case. Further information can be obtained by considering the axial velocity distribution when separation begins, i.e. when $q = q_0(\epsilon, b)$, a solution of [4.2].

5. THE AXIAL VELOCITY DISTRIBUTION BETWEEN EQUAL SPHERES

In the symmetric case ($s = 0$), the axial velocity distribution, given by [4.1] is

$$w(\xi, 0) = -1 + \cosh \frac{1}{2}\xi \sum_{1}^{\infty} (-1)^n \beta_n W_n(\xi) \qquad [5.1]$$

with $\xi_1 = \xi_2 = q$ in conditions [2.10]. These yield the equations

$$A_n \cosh(\beta_n - 1)q + C_n \cosh(\beta_n + 1)q = -\left(1 + \frac{1}{2}\epsilon\right) H_n \, e^{-\beta_n q}(\beta_n \sinh q + \cosh q)$$

$$B_n \sinh(\beta_n - 1)q + D_n \sinh(\beta_n + 1)q = \frac{1}{2}\epsilon H_n \, e^{-\beta_n q}(\beta_n \sinh q + \cosh q)$$

$$(\beta_n-1)A_n\sinh(\beta_n-1)q+(\beta_n+1)C_n\sinh(\beta_n+1)q=\left(1+\frac{1}{2}\epsilon\right)H_n\,e^{-\beta_n q}(\beta_n^2-1)\sinh q$$

$$(\beta_n-1)B_n\cosh(\beta_n-1)q+(\beta_n+1)D_n\cosh(\beta_n+1)q=-\frac{1}{2}\epsilon H_n\,e^{-\beta_n q}(\beta_n^2-1)\sinh q$$

where

$$H_n=\frac{1}{\beta_n}\left(\frac{\beta_n^2-\frac{1}{4}}{\beta_n^2-1}\right).$$

The algebra is simplified by writing W_n in the form

$$W_n(\xi)=(A_n+C_n)\cosh\beta_n\xi\cosh\xi-(A_n-C_n)\sinh\beta_n\xi\sinh\xi$$
$$+(B_n+D_n)\sinh\beta_n\xi\cosh\xi-(B_n-D_n)\cosh\beta_n\xi\sinh\xi$$

where the coefficients are given by

$$(A_n+C_n)(\sinh 2\beta_n q+\beta_n\sinh 2q)=-\left(1+\frac{1}{2}\epsilon\right)H_n(2\beta_n^2\sinh^2 q+1-e^{-2\beta_n q}+\beta_n\sinh 2q)$$

$$(A_n-C_n)(\sinh 2\beta_n q+\beta_n\sinh 2q)=-\left(1+\frac{1}{2}\epsilon\right)H_n\beta_n(\cosh 2q-e^{-2\beta_n q}+\beta_n\sinh 2q)$$

$$(B_n-D_n)(\sinh 2\beta_n q-\beta_n\sinh 2q)=\frac{1}{2}\epsilon H_n(2\beta_n^2\sinh^2 q+1+e^{-2\beta_n q}+\beta_n\sinh 2q)$$

$$(B_n-D_n)(\sinh 2\beta_n q-\beta_n\sinh 2q)=\frac{1}{2}\epsilon H_n\beta_n(\cosh 2q+e^{-2\beta_n q}+\beta_n\sinh 2q).$$

Substituting into [5.1] and proceeding like Davis & O'Neill (1977) with a Watson transformation, it eventually follows that

$$w(\xi,0)\,\mathrm{sech}\,\frac{1}{2}\xi\sim\frac{3\pi\epsilon}{16}\frac{[2(q-\xi)\cosh 2q-\sinh 2q+\sinh 2\xi]}{2q\cosh 2q-\sinh 2q}$$

$$+\epsilon\int_0^\infty\left(\frac{y^2+\frac{1}{4}}{y^2+1}\right)[(\cosh q\cos y\xi-y\sinh q\sin y\xi)y\sinh(q-\xi)$$

$$-(\cosh\xi\cos yq-y\sinh\xi\sin yq)\sin y(q-\xi)]\frac{\mathrm{sech}\,\pi y\,dy}{(\sin 2yq-y\sinh 2q)}-\pi(2+\epsilon)Re$$

$$\left\{\left(\frac{\sigma_1^2+q^2}{\sigma_1^2+4q^2}\right)\frac{\left[\frac{\sigma_1}{2q}\sin\left(\frac{\sigma_1\xi}{2q}\right)\sinh\xi(\cosh 2q-\cos\sigma_1)+\cos\left(\frac{\sigma_1\xi}{2q}\right)\cosh\xi\left(1-\cos\sigma_1\frac{\sigma_1^2\sinh^2 q}{2q^2}\right)\right]}{\cosh\left(\frac{\pi\sigma_1}{2q}\right)(\sinh 2q+2q\cos\sigma_1)}\right\}$$

$$-\pi\epsilon Im$$

$$\left\{\left(\frac{\tau_1^2+q^2}{\tau_1^2+4q^2}\right)\frac{\left[\frac{\tau_1}{2q}\cos\left(\frac{\tau_1\xi}{2q}\right)\sinh\xi(\cosh 2q+\cos\tau_1)-\sin\left(\frac{\tau_1\xi}{2q}\right)\cosh\xi\left(1+\cos\tau_1-\frac{\tau_1^2}{2q^2}\sinh^2 q\right)\right]}{\cosh\left(\frac{\pi\tau_1}{2q}\right)(2q\cos\tau_1-\sinh 2q)}\right\}$$

[5.2]

after again retaining only the dominant residue terms. In the particular case $\epsilon = 0$, [5.2] is equivalent to equation [4.18] given by Davis *et al.* (1976).

Figure 2 shows, for various ϵ, $w(\xi, 0)$ at the onset of separation on sphere I, i.e. when $q = q_0(\epsilon, 1)$. Evidently for ϵ less than approx 0.05, separation occurs due to the formation of a toroidal eddy on sphere I as happens when both spheres are at rest. Meanwhile, for greater ϵ, the streaming flow separates from sphere I when the body of fluid trapped around sphere II spreads to the surface of the other sphere.

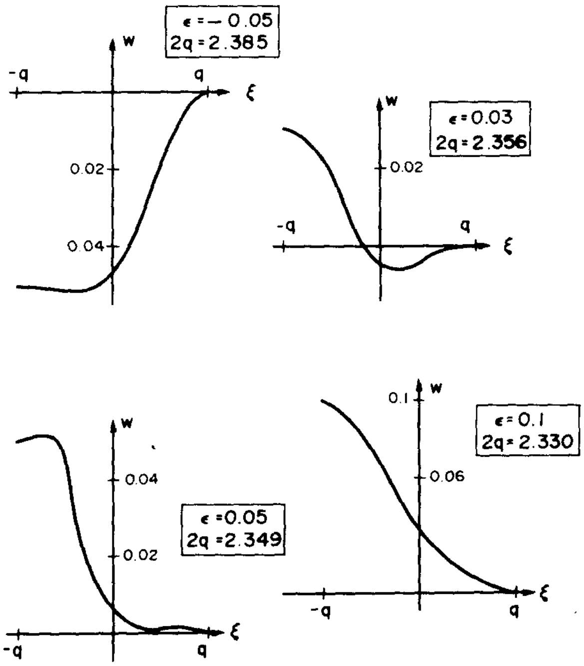

Figure 2. The axial velocity $w(\xi, 0)$ at the onset of separation on sphere I.

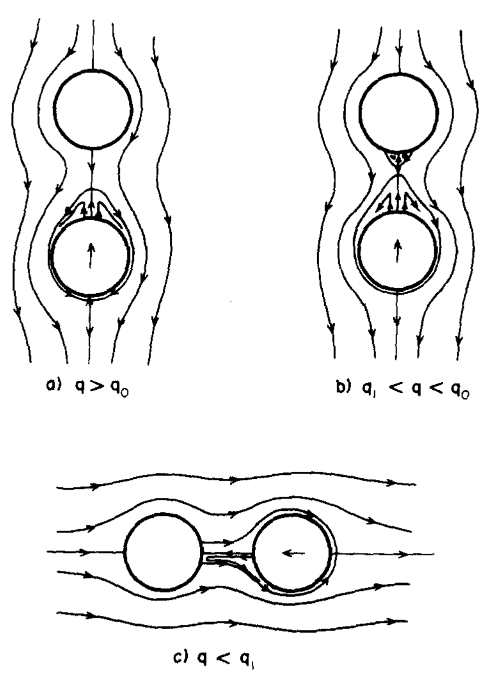

Figure 3. Typical flow patterns past equal spheres for (a) $q > q_0$, (b) $q_1 < q < q_0$ and (c) $q < q_1$.

As q is reduced for a given negative ϵ, the growth of the eddy is considerably influenced by $|\epsilon|$, being limited to within 60% of the gap between spheres at $\epsilon = -0.01$, 20% at $\epsilon = -0.03$ and 1% at $\epsilon = -0.05$.

As q is reduced for a given ϵ between 0 and 0.05, the eddy grows and joins up with the body of fluid surrounding sphere II when $q = q_1(\epsilon, 1)$, say.Typical instantaneous flow patterns are illustrated in figure 3. As ϵ increases from 0 to ~ 0.05, $2q_1(\epsilon, 1)$ increases from 2.109 (given as $2\alpha_1^+$ by Davis *et al.* (1976)) through 2.286 at $\epsilon = 0.03$ until it coincides at 2.35 with $2q_0(\epsilon, 1)$, which in the meantime has decreased from 2.367.

For ϵ greater than 0.05, the instantaneous patterns for $q >$ and $< q_0(\epsilon, 1)$ are essentially those of figure 3(a), (c) respectively.

This section has considered equal spheres in order to keep the algebra within reasonable bounds. Similar results clearly hold for $b \neq 1$, the principal conclusion being that the eddy structure present when the spheres are at rest is quickly destroyed by the introduction of relative velocity. Indeed, since the stream function changes sign on crossing from the mainstream to the primary wake and then, if it exists, to a secondary wake, it is evident from [2.8] that, even if ϵ is small enough for the quasi-static approximation to remain valid, a secondary wake cannot, for any $\epsilon > 0$, form on sphere II although it may do so on sphere I. (Similarly, for any $\epsilon < 0$, a primary wake cannot attach to sphere II). Thus, supposing that two wakes are formed in the stationary situation described by Davis *et al.* (1976), a small increase of ϵ from zero causes the primary wake to wrap around sphere II, making the secondary wake detach from this sphere.

It should be emphasised again that the flow patterns of figures 1 and 3 are instantaneous and therefore do not represent actual paths of particle motion.

REFERENCES

BRENNER, H. 1961 The slow motion of a sphere through a viscous fluid towards a plane surface. *Chem. Engng Sci.* **16**, 242–251.

DAVIS, A. M. J., O'NEILL, M. E., DORREPAAL, J. M. & RANGER, K. B. 1976 Separation from the surface of two equal spheres in Stokes flow. *J. Fluid Mech.* **77**, 625–644.

DAVIS, A. M. J. & O'NEILL, M. E. 1977 The development of viscous wakes in a Stokes flow when a particle is near a large obstacle. *Chem. Engng Sci.* **32**, 899–906.

O'NEILL, M. E. & MAJUMDAR, S. R. 1970 Asymmetrical slow viscous fluid motions caused by the translation or rotation of two spheres. Part II: Asymptotic forms of the solutions when the minimum clearance between the spheres approaches zero. *ZAMP* **21**, 180–187.

RUSHTON, E. & DAVIES, G. A. 1973 The slow unsteady settling of two fluid spheres along their line of centres. *Appl. Scient. Res.* **28**, 37–61.

WAKIYA, S. 1975 Application of bipolar coordinates to the two-dimensional creeping motion of a liquid. I. Flow over a projection or a depression on a wall. *J. Phys. Soc. Japan* **39**, 1113–1120.